Problem Set IV: Due last day of classes.

N.B.: Many of the problems are straightforward. Their purpose is to ensure you learn topics not covered in class.

1.)   Kulsrud, Chapter 6; Problem 1.

2.)   Kulsrud, Chapter 6; Problem 2.
       (N.B.: You need only obtain the expression given in the problem.)

3.)   Kulsrud, Chapter 6; Problem 4.

4.)   Determine the structure (of the layer) for a weak shock wave.

       [Hint: Use the expression for the speed of a weak shock.]

5.)   Consider a kinematic wave equation of the form:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} - v \frac{\partial^2 \rho}{\partial x^2} = -\alpha \rho^n \,,$$

where $\alpha > 0$. Take $\gamma \to 0$, and assume an initial pulse with $\rho > 0$, and $\rho = \rho(x,t)$ propagating toward $x > 0$. What is the effect of α on shock formation for $n = 1$? Explain the key competition. Under what condition will shocks form? What of $n = 2$? Why?

6.)   Consider an *oblique* MHD shock. When will the shock refract toward/away from normal? Calculate the jump ratios in both cases. What are the conditions for "switch on" ($B_{1y}=0$, $B_{2y}\neq0$) and "switch off" ($B_{1y}\neq0$, $B_{2y}=0$) shocks?

        N.B.: Read the notes and Landau and Lifshitz "Electrodynamics of Continuous Media". Pay attention to the DeHoffman-Teller trick.

7a.)   Using the same type of consideration as discussed in class, show that the decorrelation rate for a particle in an ensemble of stochastic electric fields in 1D scales as:

$$1/\tau_c \sim \left(k^2 D_{QL}\right)^{1/3}.$$

Here, D_{QL} is the quasilinear diffusion co-efficient and decorrelation is defined relative to a wave number k.

b.)   Extend the idea of integration over unperturbed orbits from 218A to show that formally the response of the distribution function to the electric field is given by

$$f_{\underset{\omega}{k}} = -\frac{q}{m} E_{\underset{\omega}{k}} \partial\langle f\rangle/\partial v \int_0^\infty d\tau\, e^{i(\omega-kv)\tau}\, e^{ik\delta x(-\tau)},$$

where $\delta x(-\tau)$ is the deviation from the unperturbed obit.

c.)   Now, assume $\delta x(-\tau)$ is produced by diffusion in velocity and thus show

$$f_{\underset{\omega}{k}} = -\frac{q}{m} E_{\underset{\omega}{k}} \frac{\partial\langle f\rangle}{\partial v} \int_0^\infty d\tau\, e^{i(\omega-kv)\tau}\, e^{-k^2 D\tau^3/3}.$$

d.)   What is the physics of the effect discussed in c.)?

e.)   Taking $\tau^3/\tau_c^3 \to \tau/\tau_c$ for convenience, what does this problem imply about resonance widths at finite amplitude? What does validity of the quasilinear calculation of D imply about the resonance width?

8.)   Consider a passive scalar, with concentration c, immersed in a turbulent flow. c obeys the equation:

$$\frac{\partial c}{\partial t} + \underline{v} \cdot \underline{\nabla} c - D\nabla^2 c = \tilde{f}_c .$$

Let c have dissipation rate α, i.e.

$$\alpha = \tilde{c}_0^2 \, v_0 / \ell_0 .$$

i.)   Calculate the $K41$ inertial range spectrum for concentration fluctuations.

ii.)   *Quantitatively* discuss what happens if

$$D \ll v \text{ (viscosity)}, \; D \gg v .$$

b.)   Consider low Mach number incompressible turbulence, with $M = v_0/c_s \ll 1$. Show

$$\ell_{diss}/\ell_{mfp} \sim M^{-1} Re^{1/4} .$$

Thus, the validity of continuum hydrodynamics gets *better* at high Re.

9.)   Consider a bounded, 2D shear flow with $V_y = V_y(x)$, with no slip boundary conditions.

a.)   Derive the linear equation governing the inviscid stability of this flow. (Hint: consider vorticity dynamics.)

b.)   Show that an inflection point (i.e. a point where $V_y''(x) = 0$) is necessary for instability. (Hint: Assume growth, and construct a complex quadratic form. What does this form imply about the growth rate?)

c.)   What happens when a β-effect is added, as in QG dynamics?

d.)   Derive the quasilinear equation for mean vorticity ω for this problem. When is this applicable?

e.)   Now, assume that fluctuations are maintained by external stirring. What can be said about the mean time asymptotic vorticity profile assuming that no slip boundary conditions apply at each boundary? Discuss the physics of your result. (Hint: Review the H-theorem for quasilinear theory.)

10.)   This problem extends Problem 9.

        Now consider the flow of Problem 3 in an MHD fluid with $\eta = \nu$, but negligible. Assume a magnetic field $\underline{B} = B_0\hat{y}$ is present.

a.)   Derive the linear eigenmode equation for the ideal MHD problem. Discuss the structure of the problem.

b.)   What is the effect of the magnetic field on linear stability? Quantify your answer.

c.)   What is the relation between vorticity flux $\langle \tilde{v}_x\tilde{\omega} \rangle$ and Reynolds stress $\langle \tilde{v}_x\tilde{v}_y \rangle$? Explain the physics behind this result.

d.)   Derive the mean vorticity and mean flow evolution equations. What is the effect of the magnetic field on momentum transport? What would happen for a strong field $\underline{B_0}$? How strong is strong? - i.e. what is the FOM?

11.)   Calculate the shock formation time for general initial conditions in nonlinear kinematic wave waves.

12.)   Calculate the R-H condition for kinematic waves.

13.)   Calculate the width of the shock layer for kinematic waves. You can leave your answer in integral form.

14.)   Consider a noisy kinematic wave system with:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(V(\rho)\rho\right) - v\partial x^2 \rho = S_0\,\delta(x-x')\delta(t-t).$$

a.)   Calculate the spectrum of $\left\langle \tilde{\rho}^2 \right\rangle$, to lowest order.

b.)   Calculate the renormalized response time for $\left\langle \tilde{\rho}^2 \right\rangle$.

c.)   What is the effect of shock onset on your result?